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A MATRIX METHOD FOR THE ANALYSIS OF BENDING VIBRATIONS, BASED ON THE DEFORMATION TECHNIQUE

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A matrix method, using a so-called deformation method for computing vibrating beams and frames of intricate configuration, is described, with construction schemes for stiffness and flexibility matrices. The system is based on an elementwise computation of strain, using flexure and torsion as system coordinates and yielding higher accuracy at less matrix rows. Variable cross sections, shearing strain, torsional inertia, etc. of two-dimensional frames and beams are covered by the method, which can be extrapolated also to three-dimensional systems under torsion and is suitable for digital computer programming. Numerical data on a conventional iteration process are given for calculating higher eigenvalues and for treating singular stiffness matrices.

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1. Introduction

For the computation of complex rodlike vibrating systems (beams and frames) methods are desirable that attack the problem by individual elements. The structure of the system of certain repetitive structural elements is to be fully covered by the procedure in such a method. In addition, the combination of the individual elements into one system must be simple from a computational

* Numbers in the margin indicate pagination in the original foreign text.

viewpoint and as schematic as possible. The more satisfactorily these requirements are met the less will be the influence of an intricate system structure on the scope and course of the computation and the more readily can such a computation be made on an electronic computer.

A method of this type, widely used in recent years - specifically in Germany - is the method of transmission matrices*. In this method, the problem is attacked from the viewpoint of differential equations as an initial-value problem, which necessitates considerable computational effort and can actually only be done by means of an electronic computer.

Another trend has developed in modern aircraft statics, where the requirement for an elementwise treatment has been made from the statics end and was finally realized by the calculus of matrices**. The static computation is followed by a determination of the elastic properties of the system in the form of a flexibility or stiffness matrix, used as the basis for calculating the vibrations. This computation itself, as a rule, is performed rather roughly by replacing the continuously distributed masses by point masses, a process which is widely used in other fields.

In this report, we use the second of the above development trends as basis for calculating the bending vibrations of beams and frames, with the purpose of correlating the static and dynamic part of the problem which, in the above-mentioned theory, had been loosely treated; we hope that, in this manner, it will be possible to approximate the continuous distribution of the stiffness as well as of the masses with the same order of magnitude of accuracy. This be-

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* A detailed presentation and literature data are given by Klotter (Bibl.1).

** A detailed list of numerous reports, although predominantly on the static aspect of the problem, are found in a report by Argyris (Bibl.2).

comes possible by using flexure and torsion as system coordinates to which the elements of the stiffness matrix as well as of the mass matrix are referred. In view of the resultant greater accuracy, matrices of a moderate number of rows, even for higher natural frequencies, will be sufficient.

The procedure for which close relations can be proved with the integral equation as well as with the calculus of variation, as the classical methods for boundary and eigenvalue problems, was found to be highly adaptable. Here, an intricate system structure with intermediary conditions has no detrimental influence, which is also true for the static indeterminacy which does not enter the calculation. In this method, it is permissible for the cross sections to be variable and even shearing strain and torsional inertia can be taken into consideration. The method, described here for beams and two-dimensional rectangular frames, can be extrapolated also to three-dimensional systems with torsional effects.

The method leads to the development of a stiffness matrix and a mass matrix of similar external structure. For the numerical treatment of this matrix eigenvalue problem, numerous methods are available today, suitable also for electronic computers, so that the problem can be considered as practically solved as soon as these two matrices have been constructed which, incidentally, can also be done automatically on the basis of the element data. Nevertheless, for the conventional iteration method in Sections 9 and 10, we will give numerical data for calculating higher eigenvalues and for treating singular stiffness matrices.

2. Coordinates

The basis of our procedure is the representation of the bending strain by

a finite number of flexures w_j and torsions $\varphi_j = w'_j$ on one beam point j each. The continuous system is coordinated with a discrete equivalent system with /202 the coordinates w_j, φ_j, which are used by us under the common denomination y_i with continuous subscripts of i = 1, 2, ..., n. These coordinates will be called system coordinates. Possible deformations such as $w_j = 0$ at an intermediate support, which vanish because of the bearing conditions involved, are not covered by y_i since they do not occur in the equations.

These coordinates y_i (displacements and torsions) form the frame of reference for the factors used in the further method, including the stiffnesses c_{ik} as well as - if they exist - the flexibilities f_{ik} inverse to these. Thus, we have the following notations:

c_{ik} = a force or a moment K_i under unit displacement or unit torsion $y_k = 1$ at $y_j = 0$ for $j \neq k$;

f_{ik} = a displacement or torsion y_i under unit load or unit moment $K_k = 1$ at $K_j = 0$ for $j \neq k$.

The symbols c_{ik} represent elements of the stiffness matrix $\mathfrak{C}$ of the system, while the symbols f_{ik} represent elements of the corresponding flexibility matrix $\mathfrak{F}$ if $\mathfrak{C}$ is nonsingular: $\mathfrak{F} = \mathfrak{C}^{-1}$. In the case of a singular $\mathfrak{C}$ (absence of linkages), no flexibility $\mathfrak{F}$ exists.

The computation with the stiffnesses c_{ik} (constraining forces under unit strain) is known as deformation method while the computation with flexibilities f_{ik} (displacements under unit forces) is known as dynamic method. For a computation method working on the elementwise principle, as conceived by us, the deformation method is definitely preferable. This is substantiated by the fact that a unit strain will manifest itself as constraining forces (except at the deformation point itself) only at directly adjacent system points, whereas

a unit load propagates as deformation over the entire system. Therefore, we will rather use the concept of stiffness in our discussion.

The selection of the strains w_j and φ_j as coordinates is of advantage from various viewpoints. By means of these two strains, the slope of the flexure $w(x)$ within a "field" between two adjacent points j and $j + 1$ can be quite accurately reproduced by a cubic interpolation polynomial which is uniquely determined by the four boundary coordinates w_j, φ_j, w_{j+1}, φ_{j+1} of the field as "supporting values" data (so-called Hermitian interpolation)*, without necessitating any coordinates located outside of the field. Any discontinuities at the field boundaries, such as an intermediate link with a jump in φ_j, thus introduce no difficulties. In addition, such an approximation $\hat{w}(x)$, composed sectionwise of cubic polynomials, represents an "admissible function" in the sense of the calculus of variation: satisfaction of all geometric boundary and intermediate conditions, continuity in $\hat{w}$ and $\hat{w}'$ (with the exception of distinct discontinuity points).

As field boundaries, we select principally all points of geometric or dynamic discontinuities such as articulations, bearings, spring-filled supports, discrete masses, cross-section discontinuities, etc. However, also mere subdivisions of longer beam sections, made for the purpose of adequate accuracy, will be counted by us as fields whose total number will be denoted by r.

For the individual field, we are using new coordinates. After selection of a coordinate system (x, y, z) for the field with the number ρ (Fig.1), the strains with fixed sign at the beginning and at the end of the field will be denoted by w_1^ρ, φ_1^ρ, and w_2^ρ, φ_2^ρ for which we again introduce the common denota-

* This interpolation has also been mentioned by Birkhoff (Bibl.3), but the concept has been treated there in a different manner. See also a report by Schaefer (Bibl.4).

tion $y_j^\rho (j = 1, 2, 3, 4)$*. These "field coordinates" y_j^ρ are correlated with the system coordinates $y_i (i = 1, 2, \ldots, n)$ in a definite manner, which is easy to comprehend in the concrete case. This correlation can be formally represented by means of so-called incidence matrices $\mathfrak{I}_\rho$, which are matrices with the elements 0, 1, or also -1, indicating whether a system coordinate y_i is "incident" on the corresponding field or not and with which sign this coordinate appears, which latter depends on the selection of the field coordinate system. Here, each incidence matrix $\mathfrak{I}_\rho$ consists of four rows corresponding to the four field coordinates y_j^ρ and of n columns corresponding to the n system coordinates y_i. At the place ji, we have 1 for the case $y_j^\rho = y_i$ and -1 for the case $y_j^\rho = -y_i$. At all other places, we have 0. If, because of bearing conditions, one $y_j^\rho = 0$, the row j will contain only zeros. It is even simpler to eliminate these zeros entirely by canceling the corresponding row and column of the two field matrices $\mathfrak{C}_\rho$ and $\mathfrak{M}_\rho$ still to be introduced. If the vector of the n system coordinates y_i is denoted by $\mathfrak{y}$ and the vector of the four field coordinates y_j^ρ by $\mathfrak{y}_\rho$, the correlation will read

$$\mathfrak{y}_\varrho = \mathfrak{I}_\varrho \mathfrak{y} . \tag{1}$$

Finally, for later use, the r field incidence matrices $\mathfrak{I}_\rho$ are combined into /203 a system incidence matrix by superposing the field strips $\mathfrak{I}_\rho$:

$$\mathfrak{I} = \begin{pmatrix} \mathfrak{I}_1 \\ \mathfrak{I}_2 \\ \vdots \\ \mathfrak{I}_r \end{pmatrix} . \tag{2}$$

Simpler and directly applicable for machine computing is the representation by an incidence Table: For each field, the indices $\pm i$ are written row by

* A confusion of the superscripts with exponents is not likely.

row in the sequence j = 1, 2, 3, 4 for which $y_j^\rho = \pm y_i$. In the case $y_j^\rho = 0$, we use 0. The individual row of this Table will be denoted as the incidence vector of the field. This vector completely determines the classification of the respective field into the system. Here, the sequence of the fields ρ as well as of the coordinates y_i can be arbitrary.

We will demonstrate the above statements on the example of a frame, in accordance with Fig.2:

	$i=$	1	2	3	4	5	
	$y_i=$	φ_0	w_1	φ_1	φ_2	φ_3	
$j=2$		1					
3			1				Field 1
4				1			
2				1			
4					1		Field 2
1		−1					
2				1			
4						1	Field 3

Incidence Matrix $\mathfrak{I}$

$j=$	1	2	3	4
$\varrho=1$	0	1	2	3
2	0	3	0	4
3	−2	4	0	5

Incidence Table

3. Construction of the Stiffness Matrix

Our method consists in the construction of two matrices referring to the discrete strains y_i: a stiffness matrix $\mathfrak{C} = (c_{ik})$ and a mass matrix $\mathfrak{M} = (m_{ik})$.

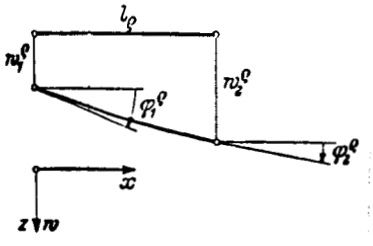

Fig.1 Field Coordinates

These are the coefficient schemes of two forms, quadratic in the coordinates y_i and representing the potential and kinetic energy of the system. Thus, both

matrices are symmetrical; $\mathfrak{M}$ is positive-definite and $\mathfrak{C}$ is definite or semidefinite. In this manner, the continuous vibration problem is coordinated with a matrix eigenvalue problem, that of the matrix pair $\mathfrak{C}$ and $\mathfrak{M}$ whose eigenvalues $\lambda =$

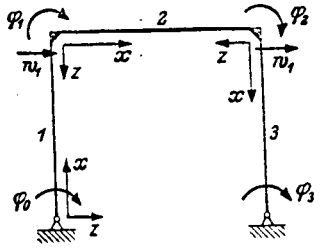

Fig.2 Construction of the Incidence Matrix for the Example of a Frame

$= \omega^2$ are real and non-negative. In accordance with our goal of an elementwise treatment of the problem, we develop both matrices from corresponding submatrices $\mathfrak{C}_\rho$ and $\mathfrak{M}_\rho$ for the individual isolated beam field. As demonstrated below,

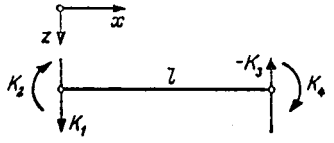

Fig.3 Calculation of Field Stiffness

these field matrices can then be used for combining the system matrices $\mathfrak{C}$ and $\mathfrak{M}$ in a simple manner, by using the incidence matrix of the system.

The construction of the stiffness matrix $\mathfrak{C}$ is a purely static problem which, incidentally, is exactly solvable. For the isolated field of the number ρ, the elements c_{ik}^{ρ} of the field matrix $\mathfrak{C}_\rho$, as constraining forces K_i for the generation of the unique strain $y_k^{\rho} = 1$ at $y_i^{\rho} = 0$ for $i \neq k$, can be determined by the conventional methods of statics. For the strain cases $k = 1$ and 2, the working theorem with the bending moment $M(x) = -K_1 x + K_2$ (Fig.3) and the

auxiliary moments $\bar{M}_1 = -x$, $\bar{M}_2 = 1$ will yield the following two equations: /204

$$\left.\begin{aligned}\int_0^{l_\varrho} \frac{M(x)\,\bar{M}_1(x)}{E\,I(x)}\,dx &= a_2 K_1 - a_1 K_2 = 1 \quad \text{or} \quad 0\text{, resp.}\\ \int_0^{l_\varrho} \frac{M(x)\,\bar{M}_2(x)}{E\,I(x)}\,dx &= -a_1 K_1 + a_0 K_2 = 0 \quad \text{or } 1\text{, resp.}\end{aligned}\right\} \tag{3}$$

Here, the coefficient a_ν are abbreviations of the following integrals:

$$a_\nu = \int_0^{l_\varrho} \frac{x^\nu}{EI(x)}\,dx = \frac{l^{\nu+1}}{E\,I_0}\int_0^1 \frac{\xi^\nu}{\alpha(\xi)}\,d\xi = \frac{l^{\nu+1}}{E\,I_0}\,\bar{a}_\nu, \tag{4}$$

where ($\nu = 0, 1, 2$) with an arbitrarily selected reference stiffness EI_0 (see Section 6). Using the coefficient matrix

$$\mathfrak{A} = \begin{pmatrix} a_2 & -a_1 \\ -a_1 & a_0 \end{pmatrix}$$

and the matrix $\mathfrak{K}$ of the constraints K_1 and K_2 for the two strain cases [the two right-hand sides of eq.(3)] - which, however, are the elements c_{ik}^{ϱ} for i, k = 1 and 2 - eq.(3) can be written in abbreviated form as $\mathfrak{AK} = \mathfrak{E}$, which yields

$$\mathfrak{K} = \mathfrak{A}^{-1} = \frac{1}{D}\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}$$

with the determinant

$$D = a_0 a_2 - a_1^2 = \frac{l^4}{(E\,I_0)^2}(\bar{a}_0\,\bar{a}_2 - \bar{a}_1^2) = \frac{l^4}{(E\,I_0)^2}\,\bar{D}. \tag{5}$$

The two other forces K_3, K_4, conversely, are obtained from the equilibrium conditions as $K_3 = -K_1$ and $K_4 = K_1 l - K_2$. This strictly defines the symmetric matrix of the field stiffnesses as

$$\mathfrak{S}_\varrho = \frac{E\,I_0}{l^3\,\bar{D}}\begin{pmatrix} \bar{a}_0 & \bar{a}_1 l & -\bar{a}_0 & (\bar{a}_0 - \bar{a}_1) l \\ \bar{a}_1 l & \bar{a}_2 l^2 & -\bar{a}_1 l & (\bar{a}_1 - \bar{a}_2) l^2 \\ -\bar{a}_0 & -\bar{a}_1 l & \bar{a}_0 & -(\bar{a}_0 - \bar{a}_1) l \\ (\bar{a}_0 - \bar{a}_1) l & (\bar{a}_1 - \bar{a}_2) l^2 & -(\bar{a}_0 - \bar{a}_1) l & (\bar{a}_0 - 2\,\bar{a}_1 + \bar{a}_2) l^2 \end{pmatrix}, \tag{6}$$

where $l = l_\varrho$ must be read everywhere. The matrix is singular, of

the rank 2 since the free field is doubly-unrestrained.

For the important special case of constant bending strength $EI = EI_\rho$, eq.(6) is transformed into

$$\mathfrak{C}_\varrho = \frac{E\,I_\varrho}{l^3}\begin{pmatrix} 12 & 6\,l & -12 & 6\,l \\ 6\,l & 4\,l^2 & -6\,l & 2\,l^2 \\ -12 & -6\,l & 12 & -6\,l \\ 6\,l & 2\,l^2 & -6\,l & 4\,l^2 \end{pmatrix} \quad \text{with } l = l_\varrho . \tag{7}$$

In the case of variable bending strength $EI(x) = \alpha(x)EI_0$, the coefficients $\bar{a}_\nu$, in accordance with eq.(4), can be calculated by numerical integration (Simpson rule).

The field stiffness $\mathfrak{C}_\rho$ is the matrix of the potential field energy (work of deformation) A_ρ, in accordance with

$$2\,A_\varrho = \sum_{i,k=1}^{4} c_{ik}^{\varrho}\, y_i^{\varrho}\, y_k^{\varrho} = \mathfrak{y}_\varrho'\,\mathfrak{C}_\varrho\,\mathfrak{y}_\varrho . \tag{8}$$

These field energies are cumulatively added to the total energy A, in which case it must only be taken into consideration that, in general, one system coordinate y_i refers to several field coordinates y_j^ρ (in the incidence Table, the same number i appears in several sites). Consequently, in addition to the summation of the expressions (8), the field coordinates must be converted into system coordinates, which can be done formally by means of the incidence matrices $\mathfrak{Z}_\rho$ in accordance with eq.(1). This will yield, for the total energy A,

$$2\,A = \sum_{\varrho=1}^{r} \mathfrak{y}'\,\mathfrak{Z}_\varrho'\,\mathfrak{C}_\varrho\,\mathfrak{Z}_\varrho\,\mathfrak{y} = \mathfrak{y}'\left(\sum_{\varrho=1}^{r} \mathfrak{Z}_\varrho'\,\mathfrak{C}_\varrho\,\mathfrak{Z}_\varrho\right)\mathfrak{y} .$$

In this case, the inner sum can be formed by combining the field stiffnesses $\mathfrak{C}_\rho$, similar to a diagonal matrix, into an auxiliary matrix /205

$$\mathfrak{C}^0 = \begin{pmatrix} \mathfrak{C}_1 & & 0 \\ & \mathfrak{C}_2 & \\ & & \ddots \\ 0 & & \mathfrak{C}_r \end{pmatrix} \tag{9}$$

and then multiplying this matrix from right to left by the total incidence matrix $\mathfrak{I}$ or $\mathfrak{I}'$. This will furnish the wanted system stiffness matrix

$$\mathfrak{C} = \mathfrak{I}' \mathfrak{C}^0 \mathfrak{I} \tag{10}$$

and thus also the total work of deformation A, from

$$2A = \mathfrak{y}' \mathfrak{C} \mathfrak{y} . \tag{11}$$

The operation (10) results in a partial overlapping of the field matrices $\mathfrak{C}_\rho$ lined up in $\mathfrak{C}^0$. For a practical execution of this calculation, see Section 8.

In the case of spring-filled supports and rotary-spring supports with the spring constants c_j and C_j respectively, the energy expressions of the form $c_j w_j^2$ or $C_j \varphi_j^2$ also must be added, which corresponds to a mere addition of the spring constants c_j or C_j to the corresponding diagonal element of the total stiffness $\mathfrak{C}$. In the case of coupling springs, acting between two coordinates y_i and y_k, the additional energy will be

$$c_j (y_i - y_k)^2 \quad \text{or} \quad C_j (y_i - y_k)^2, \text{ resp.}$$

This can be treated like an additional coupling field with the following two-row field matrix, referring to the system indices i, k:

$$c_j \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad C_j \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \text{ resp.} \tag{12}$$

To this corresponds an additional two-row incidence matrix which has the function of distributing the elements $\pm c_j$ or $\pm C_j$ over the correct places i, i; i, k; k, i; and k, k.

4. Construction of the Mass Matrix

The process of constructing the mass matrix, which is easiest to derive from the kinetic energy, is quite similar. The timewise maximum value of this

matrix, in the case of a sinusoidal vibration, will be

$$T = \frac{1}{2}\omega^2 \int \mu(x)\, w^2(x)\, dx$$

with the (generally variable) mass distribution $\mu(x)$ per unit length, to which might still be added terms of the form of

$$\frac{1}{2}\omega^2 m_j w_j^2 \quad \text{and} \quad \frac{1}{2}\omega^2 \Theta_j \varphi_j^2$$

for point masses and rotary masses which, however, are neglected for the moment. Then, we replace the unknown deformation function $w(x)$ fieldwise within each field, by a cubic interpolation polynomial of the following form:

$$\underset{(\varrho)}{\hat{w}}(x) = H_1(x)\, w_1^\varrho + H_2(x)\, \varphi_1^\varrho + H_3(x)\, w_2^\varrho + H_4(x)\, \varphi_2^\varrho ,$$

or, in abbreviated form,

$$\underset{(\varrho)}{\hat{w}}(x) = \sum_{j=1}^{4} H_j(x)\, y_j^\varrho \tag{13}$$

with the following Hermite interpolation polynomials*

$$\left.\begin{aligned} H_1(x) &= 1 - 3\xi^2 + 2\xi^3 , \\ H_2(x) &= (\xi - 2\xi^2 + \xi^3)\, l , \\ H_3(x) &= 3\xi^2 - 2\xi^3 , \\ H_4(x) &= (-\xi^2 + \xi^3)\, l , \end{aligned}\right\} \quad \text{with} \quad \xi = \frac{x}{l} . \tag{14}$$

For the field energy, this again will yield a form, quadratic in the field coordinates,

$$2\, T_\varrho = \omega^2 \sum_{i,k=1}^{4} m_{ik}^\varrho\, y_i^\varrho\, y_k^\varrho = \omega^2\, \mathfrak{y}_\varrho'\, \mathfrak{M}_\varrho\, \mathfrak{y}_\varrho$$

with the coefficients m_{ik}^ϱ which can be fieldwise calculated by means of the Hermite polynomials $H_j(x)$ and the mass distribution $\mu(x)$, yielding /206

$$m_{ik}^\varrho = \int_0^{l_\varrho} \mu(x)\, H_i(x)\, H_k(x)\, dx . \tag{15}$$

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* Basically, higher derivatives can also be used as coordinates; in this case, up to the third derivative in accordance with the order 4 of the bending differential equation. This was done by Falk (Bibl.5).

These coefficients form the elements of the four-row field mass matrix $\mathfrak{M}_\rho$ which approximates the inertia property of the ρ-th field. The integrals (15), in the case of variable mass distribution, can be interpreted by numerical integration. For the case of constant mass density $\mu = \mu_\rho = \text{const}$, these integrals can be formally calculated, yielding the following arrangement for the field matrix:

$$\mathfrak{M}_\varrho = \frac{\mu_\varrho\, l_\varrho}{420}\begin{pmatrix} 156 & 22\,l_\varrho & 54 & -13\,l_\varrho \\ 22\,l_\varrho & 4\,l_\varrho^2 & 13\,l_\varrho & -3\,l_\varrho^2 \\ 54 & 13\,l_\varrho & 156 & -22\,l_\varrho \\ -13\,l_\varrho & -3\,l_\varrho^2 & -22\,l_\varrho & 4\,l_\varrho^2 \end{pmatrix}. \tag{16}$$

By summation of the field energies to the total energy, under conversion of the field coordinates y_j^ρ to system coordinates y_i by means of the incidence matrices, the following system mass matrix will be obtained in the same manner as in the case of stiffnesses:

$$\mathfrak{M} = \mathfrak{I}'\,\mathfrak{M}^0\,\mathfrak{I} \tag{17}$$

with the auxiliary matrix

$$\mathfrak{M}^0 = \begin{pmatrix} \mathfrak{M}_1 & & & 0 \\ & \mathfrak{M}_2 & & \\ & & \ddots & \\ 0 & & & \mathfrak{M}_r \end{pmatrix}, \tag{18}$$

in which the field matrices $\mathfrak{M}_\rho$ are diagonally arranged. By the operation (17), these field matrices are made to overlap partially. The total energy will then be

$$2\,T = \omega^2\,\mathfrak{y}'\,\mathfrak{M}\,\mathfrak{y}\,. \tag{19}$$

Again, possibly present individual masses m_j and torsional inertias Θ_j must be added to the corresponding diagonal elements of $\mathfrak{M}$. Here again, it is well possible that coupling terms appear, namely, in the case of extension rods if their deformation is assumed as linear (neglecting sinusoidal longitudinal vibrations). For example, the mass of coupling springs can be taken into consid-

eration in this manner. In the case of uniform mass distribution, a coupling field with a two-row field matrix

$$\frac{1}{6} m_j \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{or} \quad \frac{1}{6} \Theta_j \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{resp.}, \tag{20}$$

is obtained, where the last term refers to coupling by a torsion rod.

5. Eigenvalue Equations, Correlation with Integral Equation and Ritz Method

From the energy theorem A + T = const, assumed for the time-variant variables $\overline{y}_i = y_i \sin \omega t$, the equations of motion are derived in the conventional manner by differentiation. This leads, for the amplitudes, to the eigenvalue equation of the matrix pair $\mathfrak{C}, \mathfrak{M}$, namely,

$$\mathfrak{C}\,\mathfrak{y} = \lambda\,\mathfrak{M}\,\mathfrak{y} \quad \text{with} \quad \lambda = \omega^2. \tag{21}$$

In eq.(21), the expression $\mathfrak{y}'\mathfrak{C}\mathfrak{y}$ is exact for the potential energy, whereas the expression for the kinetic energy $\mathfrak{y}'\mathfrak{M}\mathfrak{y}$ contains the replacement of the bending strain w(x) by the approximation $\hat{w}(x)$ constructed fieldwise of cubic polynomials; thus, also the eigenvalues λ and the eigenvectors $\mathfrak{y}$ of eq.(21) can be only approximations of the exact values, as had to be expected.

In the case of nonsingular system stiffness $\mathfrak{C}$, there exists the inverse matrix $\mathfrak{F}$ of the flexibility factors f_{ik}. On multiplying the left-hand side of eq.(21) by $\mathfrak{C}^{-1} = \mathfrak{F}$, this yields the eigenvalue equation of the matrix $\mathfrak{FM}$

$$\mathfrak{y} = \lambda\,\mathfrak{F}\,\mathfrak{M}\,\mathfrak{y} \quad \text{with} \quad \lambda = \omega^2, \tag{22}$$

which is a transform that might be of advantage for numerical purposes. However, eq.(22) represents also a direct approximation of the integral equation of the problem. This is due to the fact that, by means of the influence func-

tion f_{ix}, i.e., the deformation at the point i by the load 1 at the point x, the local strain y_i produced by inertia stresses $\omega^2\mu(x)w(x)$ can be written in the integral form

$$y_i = \omega^2 \int \mu(x)\, w(x)\, f_{ix}\, dx\,. \tag{23}$$

This exact relation directly reduces to eq.(22) and thus also to eq.(21) if, /207 first, w(x) is replaced by the polynomial approximation $\hat{w}(x)$ as had been done previously and, second, the influence function is treated in the same manner. This is possible on the basis of the well-known symmetry property $f_{ix} = f_{xi}$, where f_{xi} again represents a bending strain at the point x due to unit load or unit moment at a coordinate point i. This bending line f_{xi}, consequently, can be approximated sectionwise as w(x), by using the $f_{ki} = f_{ik}$ as supporting values. The result of the calculation is then equal to eq.(22), which gives the correlation of our method with the integral equation for the case that an influence function does exist.

Our procedure also has to do with the calculus of variation. It is known that, in this calculus, the problem of bending vibration is formulated as an extremal requirement for the regular Rayleigh quotient:

$$R[w] = \frac{\int E\, I(x)\, [w''(x)]^2\, dx}{\int \mu(x)\, w^2(x)\, dx} = \text{Extr.}, \tag{24}$$

where w(x) is to traverse the region of all admissible functions. In the case of individual springs and individual masses, the corresponding finite energy expressions enter in both numerator and denominator. The most important approximation method in the calculus of variation, namely the Ritz method, replaces the function w(x) to be varied, after selection of several fixed so-called coordinate functions $v_i(x)$ which must be "admissible" in the sense of the calculus of variation, by a linear argument.

$$\hat{w}(x) = \sum_{i=1}^{n} y_i\, v_i(x) \tag{25}$$

with still free parameters y_i that must be so defined that the extremal requirement (24) will be satisfied at least within the region of the functions covered by the argument (25).

The sectionwise cubic approximation $\hat{w}(x)$, used by us here, accurately has the form of eq.(25), the parameters y_i are our system coordinates, and the strains w_j and φ_j are at the field boundaries. However, the coordinate functions $v_i(x)$, including the derivation, constitute steady linear combinations of the Hermite polynomials $H_j(x)$. In our method, we used this approximation only for calculating the kinetic energy, i.e., for the denominator of eq.(24), whereas the potential energy was determined exactly, without using the bending slope $w(x)$, solely from the discrete strains y_i. The Ritz method is obtained if the argument (25) is also used for the numerator of eq.(24), i.e., the (linear) second derivatives $H_j''(x)$ of the Hermite polynomials (14). A fieldwise resolution will then yield, for the potential energy A_ρ, a quadratic form according to eq.(8), with the field stiffnesses $\tilde{c}^\rho_{ik}$ which, in difference to before, must now be calculated according to

$$\tilde{c}^\rho_{ik} = \int_0^{l_\rho} E\,I(x)\,H_i''(x)\,H_k''(x)\,dx \qquad (26)$$

For the special case of a sectionwise constant bending strength $EI = EI_\rho = \text{const}$, this second method leads to exactly the same values as before, namely, to the field stiffness $\mathfrak{C}_\rho$ in accordance with eq.(7). Consequently, in this case our procedure represents at the same time a Ritz method. At nonconstant bending strength, the two procedures must differ because of the fact that, in the earlier derivation of the matrix $\mathfrak{C}$, a sectionwise linear moment slope is used as basis which coincides with the sectionwise linear slope of w'' only if $EI_\rho = \text{const}$.

6. Computation with Referred Quantities

For a numerical calculation, it is suggested to work with dimensionless referred quantities, using suitably selected length, stiffness, and mass distribution

$$l_0, \quad E I_0, \quad \mu_0 \tag{27}$$

as reference quantities. Referred quantities will then be the following:

Length $$\bar{l}_\varrho = l_\varrho / l_0 \tag{28.1}$$

Stiffness $$\alpha_\varrho = E I_\varrho / E I_0 \tag{28.2}$$

Masses $$\bar{\mu}_\varrho = \mu_\varrho / \mu_0, \quad \bar{m}_i = m_i / \mu_0 l_0, \quad \bar{\Theta}_i = \Theta_i / \mu_0 l_0^3, \tag{28.3}$$

Spring constants $$\bar{c}_i = c_i l_0^3 / E I_0, \quad \bar{C}_i = C_i l / E I_0. \tag{28.4}$$

If, in addition, the angles φ and the moments M are replaced by the dimensionless quantities w_i or K_i:

$$\bar{\varphi} = l_0 \varphi, \qquad \bar{M} = M / l_0 \tag{29}$$

(in that case, $\varphi = 1/l_0$, i.e., $\bar{\varphi} = 1$ must be used as unit torsion), the /208 following dimensionless frequency parameter will replace $\lambda = \omega^2$:

$$\bar{\lambda} = \frac{\mu_0 l_0^4 \omega^2}{E I_0}. \tag{30}$$

In all our formulas, the quantities l, μ, m, c, φ, λ, etc. must be replaced by the vinculated quantities and the stiffness EI_ρ must be replaced by α_ρ.

7. Shearing Strain and Torsion Inertia

Even these two influences can be calculated in first approximation, in a rather simple manner. Both effects are of the same order of magnitude; for this reason, if at all, they should be considered together. The shearing strain due to the transverse force Q produces a difference between the cross-sectional inclination ψ and the inclination $\varphi = w'$ of the beam axis

$$w' - \psi = \frac{Q}{G F_s} \tag{31}$$

with the shear cross section F_s (for example, the web cross section in an I-beam) and the shear modulus G. The work of deformation will then be increased by a shear component A, in accordance with

$$2A_s = \int \frac{Q^2}{G F_s} dx .$$

Here, the previously used coordinates w_j, φ_j are replaced by w_j and ψ_j. This again, in calculating the stiffness matrix, will increase the coefficient a_2 by

$$b_2 = \int_0^{l_\varrho} \frac{dx}{G F_s(x)} \tag{32}$$

to

$$a_2' = a_2 + b_2 \tag{33}$$

while the denominator determinant will increase to

$$D' = a_0 a_2' - a_1^2 = D + a_0 b_2 . \tag{34}$$

At constant cross section, we have

$$a_2' = \frac{l^3}{3EI}(1 + 3\sigma) \tag{35}$$

with a shear parameter

$$\sigma = \sigma_\varrho = \frac{EI}{G F_s l^2} = \frac{EF}{G F_s}\left(\frac{i}{l}\right)^2 , \tag{36}$$

where all quantities refer to the ρ-th field. Then, the field stiffness matrix becomes

$$\mathfrak{C}_\varrho^* = \frac{EI}{l^3(1+12\sigma)} \begin{pmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2(1+3\sigma) & -6l & 2l^2(1-6\sigma) \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2(1-6\sigma) & -6l & 4l^2(1+3\sigma) \end{pmatrix}, \tag{37}$$

again with $l = l_\rho$, EI - EI_ρ, $\sigma = \sigma_\rho$, which means that the stiffness is reduced.

Compared to this, the mass matrix increases, because of the torsional component of the kinetic energy T, in accordance with

$$2T = \lambda \int_0^l \mu(x)\, w^2(x)\, dx + \lambda \int_0^l \vartheta(x)\, \psi^2(x)\, dx \tag{38}$$

with the inertia moment $\vartheta(x)$ per unit length

$$\vartheta(x) = \mu(x)\, i^2(x)\,. \tag{39}$$

At simultaneous consideration of the shearing strain, the flexure w(x) as well as the cross-sectional inclination $\psi(x)$, can be reproduced by a Hermite interpolation only in first approximation. However, numerical calculations indicate that the resultant accuracy is sufficient. First, the interpolation polynomial (13) is replaced by

$$\hat{w}(x) = H_1(x)\, w_1^\varrho + H_2(x)\, \psi_1^\varrho + H_3(x)\, w_2^\varrho + H_4(x)\, \psi_2^\varrho\,. \tag{13a}$$

Secondly, the following is substituted in eq.(36), in first approximation:

$$\psi(x) \approx \hat{w}'(x) = H_1'(x)\, w_1^\varrho + H_2'(x)\, \psi_1^\varrho + H_3'(x)\, w_2^\varrho + H_4'(x)\, \psi_2^\varrho \tag{40}$$

with the now only quadratic polynomials $H_j'(x)$. Accordingly, the elements m_{ik}^ϱ according to eq.(15) must be supplemented by the elements

$$t_{ik}^\varrho = \int_0^{l_\varrho} \vartheta(x)\, H_i'(x)\, H_k'(x)\, dx \tag{41}$$

Thus, at constant field cross section the following auxiliary matrix for torsional inertia will be obtained: /209

$$\mathfrak{T}_\varrho = \frac{\mu_\varrho\, l_\varrho}{30}\, \tau_\varrho \begin{pmatrix} 36 & 3\,l_\varrho & -36 & 3\,l_\varrho \\ 3\,l_\varrho & 4\,l_\varrho^2 & -3\,l_\varrho & -l_\varrho^2 \\ -36 & -3\,l_\varrho & 36 & -3\,l_\varrho \\ 3\,l_\varrho & -l_\varrho^2 & -3\,l_\varrho & 4\,l_\varrho^2 \end{pmatrix} \tag{42}$$

with a torsional inertia parameter of

$$\tau_\varrho = \vartheta_\varrho / \mu_\varrho\, l_\varrho^2 = (i/l)_\varrho^2\,. \tag{43}$$

Addition to the field matrix (16) then yields the modified matrix

$$\mathfrak{M}_\varrho^* = \frac{\mu\, l}{420} \begin{pmatrix} 156 + 504\,\tau & (22 + 42\,\tau)\, l & 54 - 504\,\tau & -(13 - 42\,\tau)\, l \\ (22 + 42\,\tau)\, l & (4 + 56\,\tau)\, l^2 & (13 - 42\,\tau)\, l & -(3 + 14\,\tau)\, l^2 \\ 54 - 504\,\tau & (13 - 42\,\tau)\, l & 156 + 504\,\tau & -(22 + 42\,\tau)\, l \\ -(13 - 42\,\tau)\, l & -(3 + 14\,\tau)\, l^2 & -(22 + 42\,\tau)\, l & (4 + 56\,\tau)\, l^2 \end{pmatrix}, \tag{44}$$

where, again, all field quantities refer to the ρ-th section: $\ell = \ell_\rho$, $\mu = \mu_\rho$, $\tau = \tau_\rho$.

In the differential equation for bending vibration with shearing strain and rotational inertia, at constant cross section, the two parameters σ and τ appear as the sum $\sigma + \tau$ and the product $\sigma\tau$. If the product is neglected, this will mean - in our method - that the rotational inertia must be considered in first approximation by a correspondingly increased shear parameter $\sigma^* = \sigma + \tau$. Consequently, only the stiffness matrices will change here in accordance with eq.(37), while the mass matrices can be used in the old form (16) without torsional inertias. The result of this process is shown in the last row of the brief survey given below, where the various cases are compiled for the example of a beam clamped at one end and free at the other end, at a subdivision into only two sectors of $\ell/2$ each:

	Values for $\sqrt{\lambda} \triangleq \omega$		
	Method	*Exact*	*Error in %*
$\sigma = 0.$ $\tau = 0$	3.51771	3.51602	0.05%
$\sigma = 0.10,$ $\tau = 0$	2.90806	2.89433	0.48%
$\sigma = 0.10,$ $\tau = 0.02$	2.80887	2.83383	—1.00%
$\sigma^* = 0.12$	2.83503	,,	0.04%

8. Automatic Construction of the System Matrices

By means of the incidence vector, added to each field and consisting of four signed indices

$$i_1, i_2, i_3, i_4,$$

through which the field is inserted in the system at the correct point and in the correct sense, the system matrices $\mathfrak{C}$ and $\mathfrak{M}$ can be automatically computed from the elements c^ρ_{jk} and m^ρ_{jk} of the field matrices $\mathfrak{C}_\rho$, $\mathfrak{M}_\rho$. After writing in the four indices i_j and the elements c^ρ_{jk} and m^ρ_{jk} for the ρ-th field - for which, at constant field cross section, four values c^ρ_{jk} and six values m^ρ_{jk} are required

and, at variable cross section, six values c^{ρ}_{jk} and ten values m^{ρ}_{jk} - these elements are transferred by means of the indices i_j to the correct place ($|i_j|$, $|i_k|$) of the system matrices and there added to any values already present. In the symbolic of computer calculation, the program then reads

$$\left.\begin{aligned} c_{|i_j|,|i_k|} &:= c_{|i_j|,|i_k|} \mp \operatorname{sgn}(i_j) \cdot \operatorname{sgn}(i_k) \cdot c^{\rho}_{jk}, \\ m_{|i_j|,|i_k|} &:= m_{|i_j|,|i_k|} \mp \operatorname{sgn}(i_j) \cdot \operatorname{sgn}(i_k) \cdot m^{\rho}_{jk}. \end{aligned}\right\} \tag{45}$$

For the case of $i_j = 0$ or $i_k = 0$, no calculation need be made.

9. Numerical Treatment

Of the many methods available for the treatment of the matrix eigenvalue problem (21), we are selecting the iterative calculation of eigenvalue and eigenvector as specifically advantageous for vibration problems. Of the totality of n eigenvalues λ_i of the matrix problem (21), only a certain number are in question as useful approximation values for the natural frequencies, starting with the lowest value λ_1 of the fundamental frequency and increasing to higher values λ_2, λ_3,... . At appropriate control, the iteration will yield only the numerically lowest value λ_1 in addition to the eigenvector $\mathfrak{h}_1$. The next higher values λ_2, $\mathfrak{h}_2$ can then be determined by iteration with an additional operation. As additional or auxiliary operation, we will select a method developed by Hotelling and denoted as "deflation" process, which we will modify to the problem (21) of the matrix pair $\mathfrak{M}$, $\mathfrak{C}$. At proper performance of the /210 calculation, the errors produced by the loss of significant digits can be kept to a minimum.

The desired convergence to the lowest value λ_1 (the highest reciprocal $\varkappa_1 = 1/\lambda_1$) is obtained by the iteration instruction

$$\mathfrak{C}\,\mathfrak{z}^{*}_{\nu+1} = \mathfrak{M}\,\mathfrak{z}_{\nu} = \mathfrak{w}_{\nu}, \qquad \nu = 0, 1, 2, \ldots, \tag{46}$$

based on an arbitrary vector $\mathfrak{z}_0$, supplemented by a norming instruction

$$\mathfrak{z}_{r+1} = \mathfrak{z}^*_{r+1} \frac{1}{z^*_{r+1,r}}, \tag{46a}$$

i.e., division by a certain component, for example, the r-th component which is highest in value. The iteration (46) requires the solution of a linear system of equations with fixed coefficient matrix $\mathfrak{C}$ at variable right-hand sides $\mathfrak{M}\mathfrak{z}_\nu = \mathfrak{w}$ (known as fractional iteration). In this case, a nonsingular stiffness matrix is assumed, $\det \mathfrak{C} = 0$. The case of the singular matrix $\mathfrak{C}$ must be treated separately. The method converges in the sense of

$$\left.\begin{aligned} &\mathfrak{z}_r \to \mathfrak{y}_1, \\ &\mathfrak{z}^*_{r+1} \to \varkappa_1 \mathfrak{z} \qquad (\varkappa = 1/\lambda). \end{aligned}\right\} \tag{47}$$

For deflation, i.e., for eliminating the influence of λ_1, the "left eigenvector" $\mathfrak{v}_1$ is used, which is the vector of the matrix $\mathfrak{C}\mathfrak{M}^{-1}$ transposed to $\mathfrak{M}^{-1}\mathfrak{C}$:

$$\mathfrak{C}\,\mathfrak{M}^{-1}\mathfrak{v} = \lambda\,\mathfrak{v}. \tag{48}$$

In our case - symmetry of $\mathfrak{C}$ and $\mathfrak{M}$ - the left vectors $\mathfrak{v}$ can be readily determined from the right vectors $\mathfrak{y}$ through

$$\mathfrak{v} = \mathfrak{M}\,\mathfrak{y}, \tag{49}$$

which, on substitution in eq.(48), again yields eq.(21). At different eigenvalues $\lambda_i \neq \lambda_k$, the following orthogonality exists between right and left vectors:

$$\mathfrak{M}^{(1)} = \mathfrak{M} - \frac{1}{k_1}\mathfrak{v}_1\,\mathfrak{v}_1', \tag{50}$$

If, with $\mathfrak{v}_1$, the modified matrix

$$\mathfrak{v}_i'\,\mathfrak{y}_k = \begin{cases} 0 & \text{for} \quad i \neq k, \\ k_i & \text{for} \quad i = k. \end{cases} \tag{51}$$

is formed, then this matrix, together with the unchanged stiffness $\mathfrak{C}$, will have the same eigenvalues $\varkappa_i = 1/\lambda_i$ with the exception of $\varkappa_1$ which has been transformed into zero; the eigenvectors $\mathfrak{y}_i$ remain unchanged:

$$\left.\begin{aligned} \mathfrak{M}^{(1)}\,\mathfrak{y}_1 &= \mathfrak{M}\,\mathfrak{y}_1 - \mathfrak{v}_1 = 0\,, \\ \mathfrak{M}^{(1)}\,\mathfrak{y}_i &= \mathfrak{M}\,\mathfrak{y}_i - 0 = \varkappa_i\,\mathfrak{C}\,\mathfrak{y}_i \qquad (i \neq 1)\,. \end{aligned}\right\} \tag{52}$$

which follows from the orthogonality relations (50). Now, the matrix $\mathfrak{M}^{(1)}$ need not be explicitly formed, which is of importance with respect to the possible loss of significant digits. Instead, the iteration is made on the basis of the old matrix $\mathfrak{M}$ with an auxiliary term in the form of

$$\mathfrak{C}\,\mathfrak{z}^*_{\nu+1} = \mathfrak{M}\,\mathfrak{z}_\nu - c_{1\nu}\,\mathfrak{v}_1 = \mathfrak{w}^{(1)}_\nu \tag{53}$$

with

$$c_{1\nu} = \frac{1}{k_1}\,\mathfrak{v}_1'\,\mathfrak{z}_\nu\,. \tag{54}$$

The calculation is convergent to

$$\mathfrak{z}_\nu \to \mathfrak{y}_2\,, \qquad \mathfrak{z}^*_{\nu+1} \to \varkappa_2\,\mathfrak{z}_\nu\,. \tag{55}$$

Again, norming is done according to eq.(46a). The coefficients $c_{1\nu}$, together with $\mathfrak{z}_\nu \to \mathfrak{y}_2$, tend toward zero so that the subtrahend in eq.(53) represents a small correction factor which prevents a migration toward the first eigenvector $\mathfrak{y}_1$* (also known as the J.J.Koch method).

The method can be continued; the next step operates on the matrix

$$\mathfrak{M}^{(2)} = \mathfrak{M} - \frac{1}{k_1}\,\mathfrak{v}_1\,\mathfrak{v}_1' - \frac{1}{k_2}\,\mathfrak{v}_2\,\mathfrak{v}_2' \tag{56}$$

with $k_1 = \mathfrak{v}_1'\,\mathfrak{y}_1$ but again without explicit formation of this matrix in the form of

$$\mathfrak{C}\,\mathfrak{z}^*_{\nu+1} = \mathfrak{M}\,\mathfrak{z}_\nu - c_{1\nu}\,\mathfrak{v}_1 - c_{2\nu}\,\mathfrak{v}_2 = \mathfrak{w}^{(2)}_\nu \tag{57}$$

with

$$c_{1\nu} = \frac{1}{k_1}\,\mathfrak{v}_1'\,\mathfrak{z}_\nu\,, \qquad c_{2\nu} = \frac{1}{k_2}\,\mathfrak{v}_2'\,\mathfrak{z}_\nu\,. \tag{58}$$

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Again, the two subtrahends represent small corrections that prevent a migration toward the earlier eigenvectors $\mathfrak{y}_1$, $\mathfrak{y}_2$.

10. Procedure for a Singular Stiffness Matrix

The case of the singular stiffness matrix $\mathfrak{C}$, which corresponds to the

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* A computational scheme has been given by Zurmühl (Bibl.6, 7).

absence of constraints of the vibration system, needs special treatment*. If p of the n degrees of freedom are unconstrained, the p-tuple eigenvalue $\lambda_0 = 0$ will occur. The matrix $\mathfrak{C}$ is singular, having the rank slope p so that, with respect to the eigenvector 0, exactly p linearly independent eigenvectors $\mathfrak{y}_{i0}$ exist as solutions of the homogeneous system

$$\mathfrak{C}\,\mathfrak{y}_{i0} = 0 \qquad (i = 1, 2, \ldots, p). \tag{59}$$

These yield the free motions of the system, assumed to be rigid, i.e., the motions connected with no strain energy. It is now a question to eliminate these uninteresting free solutions $\mathfrak{y}_{i0}$. This is possible by a deflation of the mass matrix $\mathfrak{M}$ to a matrix $\mathfrak{M}^{(1)}$ of the same rank slope p as $\mathfrak{C}$, again by means of the left vectors

$$\mathfrak{v}_{i0} = \mathfrak{M}\,\mathfrak{y}_{i0} \qquad (i = 1, 2, \ldots, p). \tag{60}$$

coordinated to the $\mathfrak{y}_{i0}$.

Let us imagine the two vector systems which we will combine into one matrix each:

$$\mathfrak{Y}_0 = (\mathfrak{y}_{10}, \ldots, \mathfrak{y}_{p0}) \qquad \text{and} \qquad \mathfrak{V}_0 = (\mathfrak{v}_{10}, \ldots, \mathfrak{v}_{p0}) \tag{61}$$

biorthonormed in accordance with the instruction

$$\mathfrak{v}'_{i0}\,\mathfrak{y}_{k0} = \mathfrak{y}'_{i0}\,\mathfrak{M}\,\mathfrak{y}_{k0} = \delta_{ik}, \tag{62}$$

in matrix form

$$\mathfrak{V}'_0\,\mathfrak{Y}_0 = \mathfrak{Y}'_0\,\mathfrak{V}_0 = \mathfrak{Y}'_0\,\mathfrak{M}\,\mathfrak{Y}_0 = \mathfrak{E}. \tag{62a}$$

Then, the new mass matrix will read

$$\mathfrak{M}^{(1)} = \mathfrak{M} - \mathfrak{V}_0\,\mathfrak{V}'_0. \tag{63}$$

This matrix has the same rank slope as $\mathfrak{C}$ since, because of eq.(62), the following is valid:

$$\mathfrak{M}^{(1)}\,\mathfrak{y}_{i0} = \mathfrak{M}\,\mathfrak{y}_{i0} - \mathfrak{v}_{i0} = 0. \tag{64.1}$$

However, for the remaining eigenvectors $\mathfrak{y}$, belonging to the actually interesting eigenvalues $\lambda_i \neq 0$, the following is valid because of the automatically satis-

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* More details are given elsewhere (Bibl.8).

fied orthogonality $\mathfrak{v}_{i0}' \mathfrak{y}_i = 0$:

$$\mathfrak{M}^{(1)} \mathfrak{y}_i = \mathfrak{M} \mathfrak{y}_i - 0 = \mathfrak{v}_i , \tag{64.2}$$

which means that the problem (21) can be replaced by

$$\mathfrak{C} \mathfrak{y} = \lambda \mathfrak{M}^{(1)} \mathfrak{y} . \tag{21a}$$

Reduced matrices $\hat{\mathfrak{C}}$ and $\hat{\mathfrak{M}}$ of the number of rows n - p are readily obtained by canceling the p same (but otherwise arbitrary) rows and columns in $\mathfrak{C}$ and $\mathfrak{M}^{(1)}$ which merely must be so selected that the reduced matrices are nonsingular. Let us assume that these are the p last. For this, let us imagine the matrix

$$\mathfrak{Y} = (c_1, \ldots, c_{n-p}, \mathfrak{Y}_0) ,$$

whose n - p first columns are those of the unit matrix while the p last columns are formed by the free solutions $\mathfrak{y}_{i0}$. Thus, we transform to new coordinates $\mathfrak{z}$ in accordance with

$$\mathfrak{y} = \mathfrak{Y} \mathfrak{z} .$$

Then, eq.(21a) changes to

$$\mathfrak{C} \mathfrak{Y} \mathfrak{z} = \lambda \mathfrak{M}^{(1)} \mathfrak{Y} \mathfrak{z}$$

with the matrices $\mathfrak{C}\mathfrak{Y}$ and $\mathfrak{M}^{(1)}\mathfrak{Y}$ whose n - p first columns coincide with those of $\mathfrak{C}$ and $\mathfrak{M}^{(1)}$, whereas the p last columns have become zero. If, for restoring the symmetry, we multiply from the left by $\mathfrak{y}'$ and again use $\mathfrak{y}$ instead of $\mathfrak{z}$, we will obtain

$$\hat{\mathfrak{C}} \mathfrak{y} = \lambda \hat{\mathfrak{M}} \mathfrak{y} \tag{65}$$

with the matrices

$$\hat{\mathfrak{C}} = \mathfrak{Y}' \mathfrak{C} \mathfrak{Y} \quad \text{and} \quad \hat{\mathfrak{M}} = \mathfrak{Y}' \mathfrak{M}^{(1)} \mathfrak{Y} ,$$

which is simply obtained from $\mathfrak{C}$ and $\mathfrak{M}^{(1)}$ on replacing the p last rows and /212 columns by zeros. Thus, the p last components of the vector $\mathfrak{z}$ become meaningless and, by equating them to zero, we will have $\mathfrak{y} = \mathfrak{z}$. This all comes down to the fact that all rows and columns that had become zero can be eliminated en-

tirely; eq.(65) must be read in this sense. The problem has thus been reduced to a problem of the order n - p because of the fact that the free motions, of no significance for the vibration process, have been eliminated. The same method, supplemented by a corresponding deflation of the stiffness matrix, can be used instead of the method mentioned in Section 9 for calculating the higher eigenvalues, in which case the order of the matrix is reduced each time by 1. Except, in this case, the vectors $\mathfrak{z}$ must again be converted to $\mathfrak{y}$. The process will not be further described here, because of the expected loss of significant digits.

The actual computational work required for a reduction of the matrices consists, in addition to performance of the deflation (63), in a biorthonorming of the vector systems $\mathfrak{y}_0$, $\mathfrak{V}_0$, of which - incidentally - only the system $\mathfrak{V}_0$ is required. This is so because the solutions of the homogeneous system of equations (59) occur in a non-orthonormed form

$$\mathfrak{x}_0 = (\mathfrak{x}_{10}, \mathfrak{x}_{20}, \ldots, \mathfrak{x}_{p0}), \qquad \mathfrak{M}\,\mathfrak{x}_0 = \mathfrak{U}_0 = (\mathfrak{u}_{10}, \mathfrak{u}_{20}, \ldots, \mathfrak{u}_{p0})$$

An argument

$$\mathfrak{x}_0 = \mathfrak{y}_0\,\mathfrak{R}, \qquad \mathfrak{U}_0 = \mathfrak{V}_0\,\mathfrak{R} \tag{66}$$

with a still to be determined transformation matrix $\mathfrak{R}$, taking the requirement (62a) into consideration, will lead to

$$\mathfrak{N} = \mathfrak{x}_0'\,\mathfrak{U}_0 = \mathfrak{R}'\,\mathfrak{y}_0'\,\mathfrak{V}_0\,\mathfrak{R} = \mathfrak{R}'\,\mathfrak{R}. \tag{67}$$

This can be realized by a so-called Cholesky resolution of the p-row symmetric matrix $\mathfrak{N} = \mathfrak{x}_0'\mathfrak{U}_0 = \mathfrak{x}_0'\mathfrak{M}\mathfrak{x}_0$ into the upper triangular matrix $\mathfrak{R}$ and its transpose $\mathfrak{R}'$. Since the number p of free coordinates generally is small with respect to n, this pretreatment of the problem can still be done by manual computation*.

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* For a practical execution, see for example Zurmühl (Bibl.9). The procedure given in Section 9.3 of that paper must be modified in the sense of a symmetric Cholesky resolution so as to preserve the correlation $\mathfrak{U}_0 = \mathfrak{M}\mathfrak{x}_0$ and $\mathfrak{V}_0 = \mathfrak{M}\mathfrak{y}_0$.

11. Typical Example

For the example of a story frame in Fig.4 with equal field values ℓ, μ, EI, these values are selected as reference quantities so that all referred quantities will become 1. The longitudinal extension of the rods is neglected, so that EF = ∞. Restriction to antimetric vibration states, under utilization of the symmetry, requires four coordinates φ_1, w_1, φ_2, w_2 which are given as y_1, y_2, y_3, y_4 in this sequence. After selecting the field coordinates in accordance with Fig.4, the incidence matrix and the incidence Table of the system will look as follows:

	$i=$	1	2	3	4	
		φ_1	w_1	φ_2	w_2	
$j=$	3		1			Field 1
	4	1				
	2	1				Field 2
	4	1				
	1		−1			Field 3
	2	1				
	1		1			Field 4
	2	1				
	3				1	
	4		1			
	2		1			Field 5
	4		1			
	1				−1	Field 6
	2		1			
	3		−1			
	4	1				

	$j=$	1	2	3	4
Field	$\varrho=1$	0	0	2	1
	2	0	1	0	1
	3	−2	1	0	0
	4	2	1	4	3
	5	0	3	0	3
	6	−4	3	−2	1

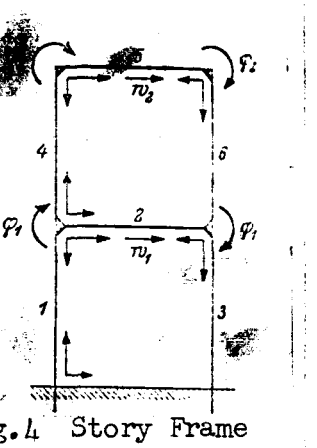

Fig.4 Story Frame

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The mass 1 of each of the two cross girders must be distributed for the bending vibration of the vertical rods as point mass, in the following manner:

$\frac{1}{4} = \frac{105}{420}$ over the field ends 1 and 6 and over the field starts 3 and 4;

$\frac{1}{2} = \frac{210}{420}$ over the field end 4 and over the field start 6.

TABLE 1

ITERATION FOR THE SECOND EIGENVALUE λ_2

	φ_1	w_1	φ_2	w_2	v_1	$v_1^{(1)}$	$v_2^{(1)}$	$v_3^{(1)}$
	18	0	—6	26	32.22001	—0.62460	23.41129	24.30805
	0	1044	—26	108	652.18748	989.84032	—1095.73518	—1093.61267
	—6	—26	10	—44	—58.24905	—0.48651	—4.31959	—4.69195
	26	108	—44	732	791.16676	—520.22699	567.52247	566.11048
	14	0	2	—6		—0.62460	23.41129	24.30805
	0	24	6	—12		989.84032	—1095.73518	—1093.61267
	— 0.1428 5714	—0.25	8.2142 8571	—2.1428 5714		—247.85736	266.26981	265.23564
	0.4285 7143	0.50	0.2608 6956	2.8695 6522		—89.69758	99.14983	98.91379
η_1:	0.4079 7920	0.5259 1605	0.1872 6424	1	1136.39976 $= \eta_1' v_1$		$\lambda_1 = 2.248\ 25$	
δ_0	1	1	—1	—1	0.0426 876			
δ_1^*	—7.96534	35.19629	—38.32827	—31.25825		—1		
δ_1	0.254 830	—1.125 984	1.226 181	1	0.0056 3108			
δ_2^*	10.56189	—38.73681	41 .42907	34.55221			—1	
δ_2	0.305 679	—1.121 109	1.199 028	1	—0.0000 0022			
δ_3^*	10.61174	—38.65265	41.28172	34.46995	c_{1r}			—1
						$\varkappa_2 = 34.476$		
η_2	0.307 960	—1.121 355	1.197 539	1		$\lambda_2 = \frac{840}{34.476} = 24.365$		
				$34.4760 = \varkappa_2$				

In this manner, the following two matrices are obtained:

$$\frac{1}{2}\mathfrak{C} = \begin{pmatrix} 14 & 0 & 2 & -6 \\ 0 & 24 & 6 & -12 \\ 2 & 6 & 10 & -6 \\ -6 & -12 & -6 & 12 \end{pmatrix},$$

$$420\,\mathfrak{M} = \begin{pmatrix} 18 & 0 & -6 & 26 \\ 0 & 1044 & -26 & 108 \\ -6 & -26 & 10 & -44 \\ 26 & 108 & -44 & 732 \end{pmatrix}.$$

In Table 1, the iteration is made for the second eigenvalue λ_2 after the iteration for λ_1 had been made. Below the matrix $\mathfrak{M}$, we have the triangle resolution (elimination) of the matrix $\mathfrak{C}$ and, following this, the elimination of the superposed right-hand sides $\mathfrak{w}_\nu$.

The two first eigenvalues

$$\lambda_1 = 2{,}24825 \qquad \text{and} \qquad \lambda_2 = 24{,}365$$

as Ritz approximations, constitute upper bounds of the exact eigenvalues. From the eigenvectors

$$\mathfrak{y}_1 = \begin{pmatrix} 0{,}40793 \\ 0{,}52592 \\ 0{,}18726 \\ 1{,}0 \end{pmatrix}$$

and

$$\mathfrak{y}_2 = \begin{pmatrix} 0{,}30796 \\ -1{,}12136 \\ 1{,}19754 \\ 1{,}0 \end{pmatrix}$$

the pertaining vibration modes of the frame can be read directly.

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